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Kawahara equation in a quarter-plane and in a finite domain * †

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ABSTRACT: This work is concerned with the existence and uniqueness of global-in-time regular solutions for the Kawahara equation posed on a quarter-plane and on a finite domain.

Key Words: Nonlinear boundary value problems; odd-order dispersive differential equations; existence and uniqueness.

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1. Introduction

This work is concerned with the existence and uniqueness of global-in-time regular solutions for the Kawahara equation posed on a quarter-plane and on a finite domain. Our study is motivated by physics and numerics: the nonlinear relation, called Kawahara equation [19], is the fifth-order dispersive-type partial differential equation describing one-dimensional propagation of small amplitude long waves in various problems of fluid dynamics and plasma physics [2,30]. This equation is also known as perturbed KdV or the special version of the Benney-Lin equation [3,4].

Methods to study initial and initial-boundary value problems for the KdV and Kawahara equations are similar, but differ in details for three types of problems, namely: the pure initial value problem (see [4,20,13,28] and the references); initial-boundary value problems posed on a finite interval (see [8,9,10,14,15,17,22,21,25,26,27]); and problems posed on quarter-planes which is the case that attracts here our

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interest. For the KdV equation there is rather developed theory of well-posedness for such problems, see, for instance, [5,6,7,11,18,29]. On the other hand, we do not know any published results dealing with well-posedness of the initial-boundary value problems for the Kawahara equation in quarter-planes and in finite domains.

The main results of this paper are the existence and uniqueness of a global regular solution to the initial-boundary value problems posed on the quarter-plane $\{x \in \mathbb{R}^+, \quad t \geq 0\}$ and on the finite domain $\{x \in (0,1), \quad t \geq 0\}$ for the nonlinear Kawahara equation. Moreover, we show that the rate of decay of the obtained solution in the quarter-plane as $x \to \infty$ does not depend on $t > 0$ and is the same as one of the initial data. Moreover, we prove that solutions to an initial-boundary value problem for the KdV equation can be approximated by corresponding solutions to initial-boundary value problems for the Kawahara equation. To prove these results we propose the transparent and constructive method of semi-discretization with respect to t which can be used for numerical simulations.

Furthermore, to obtain necessary a priory bounds in a quarter-plane, in place of usual for the KdV-type equations "artificial" weights such as $1 + x$ or $e^{x/2}$ (see citations above), we use the "natural" exponential weight e^{kx} where $k > 0$ is the decay rate of the initial data. This brings technical difficulties, but compensates in obtaining the same decay rate of the solution, while $x \to \infty$, as one of the initial data which seems to be a new qualitative property.

It should be noted also that imposed boundary conditions are reasonable both from physical and mathematical point of view, see [7,30] and comments in [15].

To prove these results, first we solve a corresponding stationary problem exploiting the method of continuation with respect to a parameter. Then we prove solvability of a linear evolution problem by the method of semi-discretization with respect to t. Using the contraction mapping arguments, we obtain a local in time regular solution to the nonlinear problem. Finally, necessary a priori estimates allow us to extend the local solution to the whole interval $t \in (0, T)$ with arbitrary $T > 0$.

2. Problems and main results

For real $T > 0$ denote $\mathbb{R}^+ = \{x \in \mathbb{R} : \quad x > 0\}$ and $Q_T = \{(x,t) \in \mathbb{R}^2 : x \in \mathbb{R}^+, \ t \in (0,T)\}$. In Q_T we consider the one-dimensional nonlinear Kawahara equation (see [19])

$$u_t - D^5u + D^3u + uDu = 0 \tag{2.1}$$

subject to initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^+, \tag{2.2}$$

$$u(0,t) = Du(0,t) = 0, \quad t \in (0,T). \tag{2.3}$$

Here and henceforth $u : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ is a unknown function, u_t denotes its partial derivative with respect to $t > 0$, $D^j = \partial^j/\partial x^j$ are the derivatives with respect to $x \in \mathbb{R}^+$ of order $j \in \mathbb{N}$ and $u_0(x) \in H^5(\mathbb{R}^+)$ is the given function satisfying

$$u_0(0) = Du_0(0) = 0. \tag{2.4}$$

Throughout the paper we adopt the usual notations $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and the inner product in $L^2(\mathbb{R}^+)$. Symbols C, C_0 and $C_i,\ i\in\mathbb{N}$, mean positive constants appearing during the text.

The main results of this article are the following.

Theorem 1 *Let $u_0\in H^5(\mathbb{R}^+)$ satisfy (2.4) and there exists a real $k>0$ such that*

$$\left(e^{kx},\left[\sum_{i=0}^{5}|D^iu_0|^2+|u_0Du_0|^2\right]\right)<\infty.$$

Then for all finite $T>0$ problem (2.1)-(2.3) has a unique regular solution

$$u\in L^\infty(0,T;H^5(\mathbb{R}^+)),$$
$$u_t\in L^\infty(0,T;L^2(\mathbb{R}^+)\cap L^2(0,T;H^2(\mathbb{R}^+))$$

such that for a.e. $t\in(0,T)$

$$\sum_{i=0}^{5}(e^{kx},|D^iu|^2)(t)+(e^{kx},u_t^2)(t)+\sum_{i=1}^{2}\int_0^t(e^{kx},|D^iu_\tau|^2)(\tau)\,d\tau$$
$$\leq C\left(e^{kx},\left[\sum_{i=0}^{5}|D^iu_0|^2+|u_0Du_0|^2\right]\right). \tag{2.5}$$

3. Stationary problem

Our purpose in this section is to solve the stationary boundary value problem

$$au-D^5u+D^3u=f(x),\quad x\in\mathbb{R}^+, \tag{3.1}$$
$$u(0)=Du(0)=0. \tag{3.2}$$

Here $a>0$ is a constant coefficient, $u:\mathbb{R}^+\to\mathbb{R}$ is an unknown bounded function, D^m denotes, as earlier, the m-th derivative with respect to x and $f(x)\in L^2(\mathbb{R}^+)$ is a given function such that

$$|f(x)|\leq C\ \mathrm{e}^{-\beta x}\text{ with }\beta>0. \tag{3.3}$$

Theorem 2 *Let $a>0$ and f satisfies (3.3). Then (3.1),(3.2) admits a unique solution $u\in H^5(\mathbb{R}^+)$ such that*

$$\|u\|_{H^5(\mathbb{R}^+)}\leq C\|f\|. \tag{3.4}$$

We start from the simpler equation

$$au-D^5u=f(x),\quad x\in\mathbb{R}^+ \tag{3.5}$$

subject to boundary data (3.2). Its general solution can be easily found by standard methods of ODE.

We solve (3.1),(3.2) making use of the method of continuation with respect to a parameter for the operator equation

$$A_\lambda u \equiv au - D^5u + \lambda D^3 u = f(x), \tag{3.6}$$

where

$$\Lambda = \Big\{\lambda \in [0,1] : \quad \{0\},\{1\} \in \Lambda\Big\}.$$

4. Linear evolution problem

Here we consider the following linear initial-boundary value problem:

$$u_t - D^5u + D^3u = f(x,t), \quad (x,t) \in Q_T; \tag{4.1}$$

$$u(0,t) = Du(0,t) = 0, \quad t \in (0,T); \tag{4.2}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^+; \tag{4.3}$$

where

$$u_0 \in H^5(\mathbb{R}^+), \quad u_0(0) = Du_0(0) = 0, \quad f, f_t \in C\left(0,T; L^2(\mathbb{R}^+)\right) \tag{4.4}$$

and

$$\left(e^{kx}, \sum_{i=0}^{5} |D^i u_0|^2\right) + \int_0^T \Big[(e^{kx}, f^2)(t) + (e^{kx}, f_t^2)(t)\Big]\, dt < \infty. \tag{4.5}$$

To aboard (4.1)-(4.3) we exploit the method of semi-discretization [24]. Define

$$h = \frac{T}{N}, \quad N \in \mathbb{N} \quad \text{and}$$

$$u^n(x) = u(x, nh), \quad n = 0, \dots, N, \quad \text{with} \quad u^0(x) = u_0(x).$$

Furthermore,

$$u_h^n = \frac{u^n - u^{n-1}}{h}, \quad n = 1, \dots, N, \quad \text{and}$$

$$u_h^0 \equiv u_t(x,0) = f(x,0) - D^3u_0(x) + D^5u_0(x). \tag{4.6}$$

We approximate (4.1)-(4.3) by the following system:

$$Lu^n \equiv \frac{u^n}{h} + D^3u^n - D^5u^n = f^{n-1} + \frac{u^{n-1}}{h}, \quad x \in \mathbb{R}^+; \tag{4.7}$$

$$u^n(0) = Du^n(0) = 0, \quad n = 1, \dots, N; \tag{4.8}$$

$$u^0(x) = u_0(x), \quad x \in \mathbb{R}^+. \tag{4.9}$$

Due to results on solvability of the boundary value problem for the stationary equation, we can prove necessary a priori estimates for u^n which imply the following

Theorem 3 *Let $u_0(x)$ and $f(x,t)$ satisfy (4.4) and (4.5). Then (4.1)-(4.3) admits a unique solution*

$$u \in L^\infty(0,T;H^5(\mathbb{R}^+)),$$
$$u_t \in L^\infty(0,T;L^2(\mathbb{R}^+) \cap L^2(0,T;H^2(\mathbb{R}^+)),$$

such that

$$\sup_{t\in(0,T)} \left\{(e^{kx},u^2)(t) + (e^{kx},u_t^2)(t)\right\}$$
$$+ \int_0^T \sum_{i=1}^2 \left\{(e^{kx},|D^iu|^2)(t) + (e^{kx},|D^iu_t|^2)(t)\right\} dt < \infty.$$

5. Nonlinear problem. Local solutions.

Using the contraction mapping principle, we prove the existence and uniqueness of local regular solutions to the following nonlinear problem:

$$u_t - D^5u + D^3u = -uDu, \quad (x,t) \in Q_T; \tag{5.1}$$
$$u(0,t) = Du(0,t) = 0, \quad t \in (0,T); \tag{5.2}$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^+. \tag{5.3}$$

Theorem 4 *Let $u_0(x) \in H^5(\mathbb{R}^+)$ satisfy (2.4) and*

$$\sum_{i=0}^5 \left(e^{kx},|D^iu_0|^2\right) + (e^{kx},|u_0Du_0|^2) < \infty.$$

Then there exists a real $T > 0$ such that (5.1)-(5.3) possesses a unique regular solution in Q_T and

$$\sup_{t\in(0,T)} \left\{(e^{kx},u^2)(t) + (e^{kx},u_t^2)(t)\right\}$$
$$+ \sum_{i=1}^2 \int_0^T \left\{(e^{kx},|D^iu|^2)(t) + (e^{kx},|D^iu_t|^2)(t)\right\} dt$$
$$\leq C(T,k)\left[\sum_{i=0}^5 (e^{kx},D^iu_0^2) + (e^{kx},|u_0Du_0|^2)\right]$$

6. Global solutions

A priori estimates uniform in $t \in (0,T)$ now allow us to extend the obtained local solution to the whole $(0,T)$ with arbitrary fixed $T > 0$, hence to prove Theorem 1.

Remark. Making use of the approach exploited to prove Theorem 1 and results from [15], we can solve the Kawahara equation posed on a finite interval:

$$u_t + uDu - D^5u + D^3u = 0, \quad (x,t) \in (0,1)\times(0,T); \tag{6.1}$$
$$u(0,t) = Du(0,t) = u(1,t) = Du(1,t) = D^2u(1,t) = 0, \ t > 0; \tag{6.2}$$
$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{6.3}$$

The following assertions are valid.

Theorem 5 *Let $u_0 \in H^5(0,1)$ and satisfies the consistency conditions related to (6.2). Then for all finite $T > 0$ there exists a unique regular solution to (6.1)-(6.3)*

$$u \in L^\infty(0,T;H^5(0,1)),$$
$$u_t \in L^\infty(0,T;L^2(0,1) \cap L^2(0,T;H^2(0,1)),$$

such that for a.e. $t \in (0,T)$

$$\|u\|^2_{H^5(0,1)}(t) + \|u_t\|^2(t) + \sum_{i=1}^{2} \int_0^T \left(\|D^i u\|^2(t) + \|D^i u_t\|^2(t)\right) dt$$
$$\leq C\|u_0\|^2_{H^5(0,1)},$$

where all the norms $\|\cdot\|$ are in $L^2(0,1)$.

Theorem 6 *Let*

$$11 - \frac{2}{3}\|u_0\| = \kappa > 0.$$

Then for all $t > 0$ the regular solution given by Theorem 5 satisfies the following inequality

$$\|u\|^2(t) \leq 4\|u_0\|^2 e^{-\kappa t}. \tag{6.4}$$

To formulate the next theorem, for real $\mu > 0$ we consider in Q_T the following problems:

$$u^\mu_t + u^\mu D u^\mu + D^3 u^\mu - \mu D^5 u^\mu = 0, \quad (x,t) \in Q_T; \tag{6.5}$$
$$D^i u^\mu(0,t) = D^i u^\mu(1,t) = D^2 u^\mu(1,t) = 0, \quad i = 0,1; \quad t \in (0,T); \tag{6.6}$$
$$u^\mu(x,0) = u^m_0(x), \quad m \in \mathbb{N}; \quad x \in (0,1) \tag{6.7}$$

and

$$u_t + uDu + D^3 u = 0, \quad (x,t) \in Q_T; \tag{6.8}$$
$$u(0,t) = u(1,t) = Du(1,t) = 0, \quad t \in (0,T); \tag{6.9}$$
$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{6.10}$$

Theorem 7 *Let $u^m_0 \in H^5(0,1)$ and $u_0 \in H^3(0,1)$ satisfy the consistency conditions related to (6.6) and (6.9) correspondingly. Suppose*

$$\|u^m_0 - u_0\|_{H^3(0,1)} \to 0 \quad as \quad m \to \infty.$$

Then for all finite $T > 0$ there exists a unique solution $u(x,t)$ to (6.8)-(6.10) such that

$$u \in L^\infty(0,T;H^3(0,1)) \cap L^2(0,T;H^4(0,1)),$$
$$u_t \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^1(0,1)).$$

Moreover, if $\mu \to 0$, and $m \to \infty$, then

$$u^\mu \to u \ \ *-weak \ in \ L^\infty(0,T;L^2(0,1)) \ and \ weakly \ in \ L^2(0,T;H^1(0,1)),$$
$$u_t^\mu \to u_t \ \ *-weak \ in \ L^\infty(0,T;L^2(0,1)) \ and \ weakly \ in \ L^2(0,T;H^1(0,1)).$$

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